

The Effects of Symmetries on Quantum Fidelity Decay

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We explore the effect of a system's symmetries on fidelity decay behavior. Chaos-like exponential fidelity decay behavior occurs in non-chaotic systems when the system possesses symmetries and the applied perturbation is not tied to a classical parameter. Similar systems without symmetries exhibit faster-than-exponential decay under the same type of perturbation. This counter-intuitive result, that extra symmetries cause the system to behave in a chaotic fashion, may have important ramifications for quantum error correction.

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Fidelity decay as a possible signature of quantum chaos was introduced by Peres [1, 2] while exploring irreversibility in quantum mechanical systems. Though the overlap between two initial states undergoing equivalent evolution remains constant with time, it decreases with time if the Hamiltonian effecting one of the systems is slightly perturbed. The behavior of the decrease in overlap, or fidelity, depends on whether the evolution is the quantum analog of a chaotic or non-chaotic classical system.

Further explorations have distinguished realms of fidelity decay behavior based on the chaoticity of the corresponding classical system, perturbation strength and type, and the initial system state. Weak perturbations of the Hamiltonian, such that perturbation theory is valid, exhibit a Gaussian fidelity decay [1, 3]. Stronger perturbations, in the Fermi golden rule (FGR) regime, exhibit exponential fidelity decay with a rate determined by the strength and type of perturbation [3, 4, 5, 6]. The rate of this exponential generally increases as the square of the perturbation strength [3] and may saturate at the underlying classical systems' Lyapunov exponent [7, 8] or the bandwidth of the Hamiltonian [3].

The fidelity decay of classical-like coherent states for quantum analogs of non-chaotic systems [4, 9, 10] can be Gaussian [4, 11], faster than the chaotic system exponential, or power-law [12] depending on the initial state [9] and the effect of the perturbation on the classical orbits on which the state is centered [13, 14]. The fidelity decay of systems [15], states [16], and perturbation strengths [17] at regime edges have also been explored.

The quantum fidelity decay of an initial state $|\psi_i\rangle$ is

$$F(t) = |\langle \psi_i | U^{-t} U_p^t | \psi_i \rangle|^2 \quad (1)$$

where U is the unperturbed evolution, $U_p = U e^{-i\epsilon V}$ is the perturbed evolution, ϵ is the perturbation strength,

and V is the perturbation Hamiltonian. For chaotic systems, random matrix theory (RMT) gives the FGR exponential fidelity decay rate Γ as a function of ϵ and the perturbation Hamiltonian eigenvalues λ_i [6]:

$$\Gamma = \epsilon^2 \overline{\lambda^2}, \quad (2)$$

where $\overline{\lambda^2} = N^{-1} \sum_i \lambda_i^2$ and N is the Hilbert space dimension.

Fidelity decay is related by a Fourier transform to the local density of states (LDOS) [3, 18] $\eta(\Delta\phi) = |\langle v_m | v'_n \rangle|^2$, where $\Delta\phi = \phi_m - \phi'_n$, is the difference between unperturbed and perturbed eigenangles given by the eigenvalue equations: $U|v_m\rangle = \exp(-i\phi_m)|v_m\rangle$ and $U_p|v'_n\rangle = \exp(-i\phi'_n)|v'_n\rangle$. The LDOS provides a measure of how local is the perturbation. For complex systems the LDOS is typically Lorentzian [19], the perturbation transfers probability to far reaches of the system basis. The Lorentzian width Γ gives the exponential chaotic fidelity decay rate.

Recently Rossini, Benenti, and Casati (RBC) [20] reported numerical evidence of exponential fidelity decay in non-chaotic systems due to 'quantum' perturbations (though the rate may not be that expected for chaotic systems). Quantum perturbations are not tied to a classical system parameter and are applied multiple times during each map iteration, i.e. after each basic gate in the simulation of the dynamics on a quantum computer. The cause of this behavior, RBC explain, is the non-locality of such errors which allow for direct transfer of probability over a large distance in phase space. Meaning, the LDOS has significant amplitude even for large $\Delta\phi$. This result is of particular importance for suggested exploitations of the slower exponential decay to stabilize quantum computation [21, 22], and for fidelity decay studies on a quantum computer [6, 20, 23, 24].

In this paper we numerically demonstrate that perturbations not tied to a classical parameter of non-chaotic systems can exhibit exponential decay even when applied after every map iteration, as generally done in fidelity decay studies. We choose this perturbation application scheme because 1) there is no unique gate sequence for

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implementing an operator 2) some of the operators we discuss cannot be implemented efficiently on a quantum computer 3) perhaps most importantly, when the perturbation is applied at numerous points during the map iteration the composite effect on the map is likely random. Thus, the exponential decay is due to the basis of the perturbation as found in [6]. By applying the perturbation every map iteration we show that the decay behavior is due to a different phenomenon: symmetries in the system Hamiltonian. When no symmetries are present the perturbation is local, the LDOS is Gaussian, and the fidelity decay is faster than exponential even when the perturbation is not tied to a classical system parameter. We emphasize the counter-intuitive nature of this claim, namely, by adding *symmetry*, the system's fidelity behavior behaves in a more *chaotic* fashion.

Our numerical study centers around the quantum kicked top (QKT), a system used in many studies of quantum chaos in general [25] and fidelity decay in particular [2, 3, 4, 6]. The QKT [26] Floquet operator is

$$U_{QKT} = e^{-i\pi J_y/2} e^{-ikJ_z^2/2J}, \quad (3)$$

where J is the angular momentum of the top and \vec{J} are irreducible angular momentum operators. The Hilbert space dimension of the top is $N = 2J + 1$ and the representation is such that J_z is diagonal. The chaoticity of the QKT depends on the kick strength, k . The QKT is non-chaotic for $k \lesssim 2.7$, has chaotic and non-chaotic regions for $2.7 \lesssim k \lesssim 4.2$, and is fully chaotic for $k \gtrsim 4.2$ [3]. The QKT has different symmetry sectors based on its angular momentum. For all J the QKT has conserved parity with respect to 180° rotations about y . For even J the subspace even with respect to y has conserved parity with respect to 180° rotations about x .

In his original fidelity decay studies, Peres was careful to account for the symmetries of the QKT [2] by block diagonalizing and performing simulations using only one block. While some later authors have done the same [4], others [3, 14] have used the complete QKT without apparent discrepancies. Here we show that the presence of symmetries in a quantum system can dramatically change the fidelity decay behavior.

We choose a ‘quantum’ perturbation relevant to quantum control studies

$$U_p = \Pi_{j=1}^{n_q} \exp(-i\epsilon\sigma_z^j/2), \quad (4)$$

where n_q is the number of qubits and σ_z is the Pauli spin matrix. This corresponds to a collective qubit rotation about the z -axis by angle ϵ and is a model of coherent far-field errors [27]. We apply the perturbation only after a complete map iteration.

Figure 1 shows the fidelity decay of the QKT for different values of k with the above perturbation, $\epsilon = .2$, and random initial states. Values of k are shown for regular, mixed, and chaotic QKT but the fidelity decay is

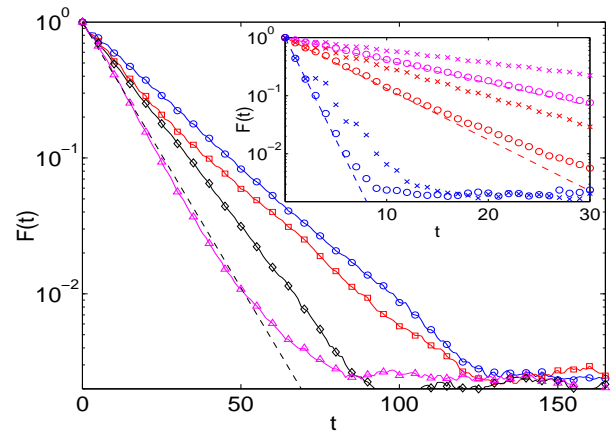


FIG. 1: (Color online) Fidelity decay for QKT with $J = 255.5$, and $k = 1$ (\circ), 3 (\square), 5 (\diamond), and 7 (\triangle) for the collective bit z rotation with $\epsilon = .2$. Even the non-chaotic QKT decays are exponential though they do not follow the random matrix theory rate given in Eq. 2 (dashed line). The inset shows the fidelity decay for $k = 1$ (\times) and 10 (\circ) for perturbation strengths $\epsilon = .2, .3$, and $.6$ (top to bottom). As the perturbation strengthens, the invariant subspaces are overwhelmed and the system behaves as if there were no symmetries. Hence, the non-chaotic QKT reverts to non-exponential fidelity decay. RMT predictions are also shown (dashed lines). All plots are averaged over 100 random initial states.

always exponential. This tells us that the fidelity decay due to a quantum error is exponential independent of the chaoticity of the system *even* when applied after every map iteration. Here there is no concern that the system has random eigenvectors in the basis of the perturbation since the perturbation is diagonal (and in this basis the regular QKT eigenvectors are not random). Initial angular momentum coherent states for the QKT and random states for the quantum Harper’s map give similar results.

Exponential fidelity decay due to quantum perturbations is not universal. We demonstrate this via unitary matrices of the interpolating ensembles [28], which are intermediate between random, $\delta = 1$, and diagonal with Poissonian distributed eigenangles $\delta = 0$. Matrix statistics for these ensembles transition smoothly between the two limits [28, 29]. There is no known way of efficiently implementing such matrices on a quantum computer. For our purposes matrices with $\delta < 1$ provide models of non-chaotic operators that have no classical analog. Any perturbation of these operators must be quantum as, a priori, the operators have no classical parameters. Fig. 2 demonstrates that the $\delta \neq 1$ fidelity decay for the perturbation of Eq. 4, $\epsilon = .3$, is *not* exponential.

An explanation for the above discrepancies is the presence of symmetries. Non-chaotic systems containing symmetries, and thus invariant subspaces, exhibit chaos-like exponential fidelity decay, while non-chaotic systems without symmetries exhibit faster-than-exponential decay. This result is surprising. A priori we would expect

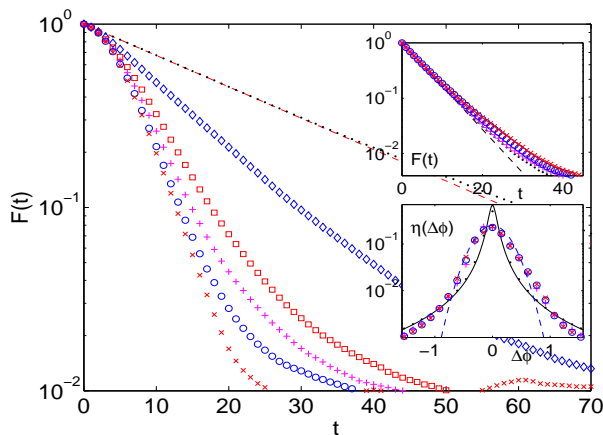


FIG. 2: (Color online) Fidelity decay for matrices of the interpolating ensemble $\delta = 1$ (\cdot), $.98$ (\diamond), $.94$ (\square), $.9$ ($+$), $.8$ (\circ), $.7$ (\times) for the collective qubit z rotation with $N = 128$ and $\epsilon = .3$. The fidelity for $\delta < .7$ matrices has the same behavior as those with $\delta = .7$. For $\delta \neq 1$ the fidelity decay is faster than exponential even though the perturbation is not tied to a classical parameter (the matrices have no classical analogs). As $\delta \rightarrow 1$ the matrices become random and the decay rate approaches the RMT rate (dashed line). All plots average over 10 operators with 100 random initial states per operator. The lower inset shows the local density of states for an average of 50 $\delta = 1, .9, .8, .7$ operators, compared to the Lorentzian expected for chaotic systems (solid line), and a Gaussian (dashed line). The Gaussian like LDOS for non-chaotic systems shows that the perturbation is localized. The upper inset shows the fidelity decay for $\epsilon = .3$, $N = 256$ block diagonal operators in which the two $N/2 = 128$ blocks are different interpolating ensemble matrices of the same δ . Changing bases and having the perturbation break the symmetry of the operator, brings the fidelity decay close to the RMT prediction (dashed line). As δ is decreased, the fidelity decay becomes slightly slower than exponential.

systems with less symmetries to be more chaos-like. Below we provide further numerical evidence demonstrating this behavior: one QKT symmetry sector and interpolating ensemble matrices with added symmetries.

Following Peres [2] we block diagonalize an even J QKT and keep one block as the system. Using the perturbation of Eq. 4, the fidelity decay for the block is faster-than-exponential for non-chaotic values of k and exponential for chaotic values of k , as shown in Fig. 3. This is in contrast to the exponential decay seen for all k values when using the complete QKT. Thus, by *removing* the symmetry, the system deviates from the expected RMT behavior. In addition, for chaotic values of k , the exponential decay rate is not the one given by Eq. 2, again in contrast to the full QKT where the decay rate was exact.

We expect the fidelity decay of systems with symmetries and extremely strong perturbations, such that the symmetries are overwhelmed, to revert to non-exponential. This occurs in the full QKT with $\epsilon = .6$,

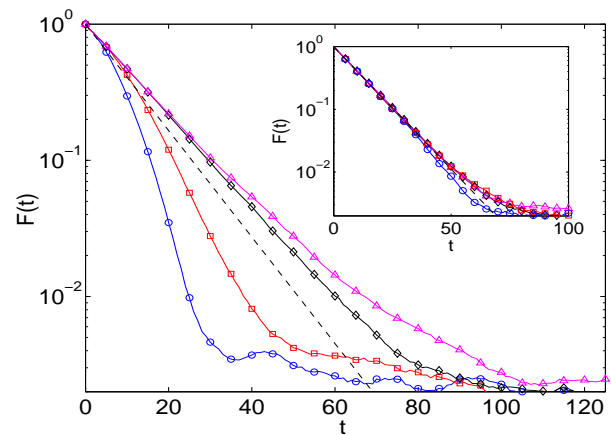


FIG. 3: (Color online) Fidelity decay for the QKT subspace even with respect to 180° rotations about y and odd with respect to 180° rotations about x , $J = 1024$, $N = 512$, $k = 1$ (\circ), 3 (\square), 5 (\diamond), and 7 (\triangle) for the collective bit z rotation with $\epsilon = .2$. Though the perturbation is not attached to a classical parameter the fidelity decay for non-chaotic QKTs are non-exponential. The chaotic QKT exponential decay does not follow the RMT rate, $\Gamma = .45$ (dashed line, actual rate is $\Gamma \simeq .405$). The inset shows the fidelity decay for the same parameters with the perturbation transformed into a random basis. The decay is exponential for all values of k at the RMT rate.

Fig. 1 inset.

A symmetry can be added to interpolating ensemble matrices by using two matrices of equal δ and Hilbert dimension $N/2$ as diagonal blocks of a $N \times N$ operator. The perturbation of Eq. 4 does not cause mixing between the blocks so we transform into a new basis. The first transformation we apply takes the perturbation to a collective qubit x rotation, σ_x replaces σ_z in Eq. 4. With this the fidelity decay is exponential at the RMT rate even for low δ blocks. However, this exponential decay is not due to the broken symmetry. Rather, the matrices of the interpolating ensembles, though not random with respect to the σ_z collective qubit perturbation, are random in the eigenbasis of the σ_x collective qubit perturbation. As explained in [6] (and shown in Fig. 3 for the QKT without symmetry), when the system is random in the perturbation basis, the fidelity decay is exponential.

A transformation which does not induce randomness in a block diagonal operator with interpolating ensemble blocks is a modified version of the transformation matrix to block diagonalize the even J QKT [2, 30]. With this transformation the fidelity decay is exponential for $\delta \neq 1$ blocks and becomes slightly slower than exponential for lower values of δ , Fig. 2. Thus, by *adding* a symmetry to the system we have slowed the fidelity decay and made its behavior more chaos-like.

The above simulations demonstrate that perturbations of non-chaotic systems, even ones not attached to a classical system parameter, tend to be local. However, when

the perturbation breaks a symmetry there are long range effects similar to those of chaotic systems. This process is different than that described in Ref. [6] where the exponential decay is linked to the RMT statistics of the system eigenvectors in the perturbation basis.

Coherent far-field errors, such as Eq. 4, are the subject of many theoretical and experimental quantum error-correction codes and encodings [27, 31]. Our results suggest that this error may cause only exponential fidelity decay even when a quantum computer is simulating non-chaotic evolution. This provides a novel way of protecting against these noise operators, *add a symmetry to the system*. Such symmetries already exist in encoded qubits, where specific states of a multi-qubit system are used as a logical qubit. For example, logical qubits in quantum dots that are encoded such that the exchange coupling is universal [32, 33] are symmetric with respect to the S_z angular momentum operator. In general, by enforcing all reasonable decoherence mechanisms to break a symmetry, decoherence may cause an exponential, rather than Gaussian decay.

Exponential decay occurs in many physical systems not generally regarded as chaotic, including relaxation phenomena and Fermi Golden Rule calculations. We suggest, without proof, that symmetry may add insight into the preponderance of exponential decay laws in nature. Lack of symmetry may explain why some systems, such as magnetic relaxation in single molecular magnets [34], decay non-exponentially.

In conclusion, we have studied the effect of symmetries on fidelity decay behavior. When the perturbation is not tied to a classical parameter of the system, as would likely arise in quantum computers, the presence or lack of symmetries strongly affects the fidelity decay behavior. Surprisingly, the presence of symmetries forces the system into a more chaos-like behavior, exponential decay, while lack of symmetries causes deviations from RMT predictions and a faster-than-exponential decay. Building symmetries into quantum computers, as done when encoded qubits, can cause decoherent processes to affect the system with the less-damaging exponential decay.

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